Discrete vortex solitons

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Localized states in the discrete two-dimensional (2D) nonlinear Schrödinger equation is found: vortex solitons with an integer vorticity S. While Hamiltonian lattices do not conserve angular momentum or the topological invariant related to it, we demonstrate that the soliton's vorticity may be conserved as a dynamical invariant. Linear stability analysis and direct simulations concur in showing that fundamental vortex solitons, with $S=1$, are stable if the intersite coupling C is smaller than some critical value $C_{\rm cr}^{(1)}$. At $C>C_{\rm cr}^{(1)}$, an instability sets in through a quartet of complex eigenvalues appearing in the linearized equations. Direct simulations reveal that an unstable vortex soliton with $S=1$ first splits into two usual solitons with $S=0$ (in accordance with the prediction of the linear analysis), but then an instability-induced spontaneous symmetry breaking takes place: one of the secondary solitons with $S=0$ decays into radiation, while the other one survives. We demonstrate that the usual ($S=0$) 2D solitons in the model become unstable, at $C>C_{\rm cr}^{(0)}\approx 2.46C_{\rm cr}^{(1)}$, in a different way, via a pair of imaginary eigenvalues ω which bifurcate into instability through $\omega=0$. Except for the lower-energy $S=1$ solitons that are centered on a site, we also construct ones which are centered between lattice sites which, however, have higher energy than the former. Vortex solitons with $S=2$ are found too, but they are always unstable. Solitons with $S=1$ and $S=0$ can form stable bound states.

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Nonlinear lattice equations naturally arise as models of various physical systems, being also an object of interest in their own right, as an important class of nonlinear dynamical systems. A fundamental lattice model is represented by the discrete nonlinear Schrödinger (DNLS) equation with cubic on-site nonlinearity, which finds its most straightforward applications, both theoretical [1] and experimental [2], to transverse dynamics in arrays of optical waveguides (fibers) with Kerr nonlinearity. In this work, we are dealing with the two-dimensional (2D) self-focusing DNLS equation for complex variables $u_{m,n}$:

$$i\dot{u}_{m,n}+C(u_{m+1,n}+u_{m,n+1}+u_{m,n-1}+u_{m-1,n}-4u_{m,n}) +|u_{m,n}|^2u_{m,n}=0, \tag{1}$$

C being a real coupling constant. As has been recently demonstrated (see Refs. [3,4] and references therein), this and similar models support stable solitons (localized stationary states) of the form $u_{m,n}(t)=\exp(i\Lambda t)U_{m,n}$, with $U_{m,n}$ determined by a stationary version of Eq. (1),

$$C(U_{m+1,n}+U_{m,n+1}+U_{m,n-1}+U_{m-1,n}-4U_{m,n}) +|U_{m,n}|^2U_{m,n}=\Lambda U_{m,n}. \tag{2}$$

In the soliton solution, $U_{m,n}$ vanish as $[(m^2+n^2)]^{1/4}\exp[-\sqrt{(\Lambda/C)(m^2+n^2)}]$ as $|m|,|n|\to\infty$, which is suggested by analogy with the continuum limit (the asymptotic expression for the far tail of the soliton must always have a quasicontinuous form). Obviously, the coupling constant C may be set equal to 1 by means of a rescaling. However, we prefer to keep C as a control parameter, fixing Λ to a constant value; unless indicated otherwise, $\Lambda\equiv 0.32$ for the numerical results given below. An advantage of this choice is that it allows us to investigate the crossover to the continuous limit, taking $C\to\infty$.

In many respects, the 2D lattice solitons found in Refs. [3,4] are similar to their earlier studied 1D counterparts [1,2,5]. This similarity is in drastic contrast with what is known about the continuous NLS equation: in that case, the commonly known 1D solitons are stable, while axisymmetric 2D solitons are not, as the 2D self-focusing NLS equation with cubic nonlinearity is subject to wave collapse [6]. However, the continuous 2D NLS equation can be easily modified, by adding a quintic self-defocusing nonlinear term, which makes the axisymmetric solitons in the corresponding *cubic-quintic* (CQ) model stable [7].

An objective of this work is to produce another type of stable 2D solitons in the DNLS model, viz., a discrete *vortex soliton* (VS), which is an exponentially localized bright soliton with a vortex nested inside it. We stress its difference from discrete analogs of the well-known optical vortices [8], which exist on a finite background, i.e., are solitons of the dark type. Bright VS's (alias vortex rings) are known in various continuum models. In the 2D cubic NLS equation they are definitely unstable, as well as the usual (nonvortex) solitons, due to collapse. In the 2D NLS equation of the CQ type, a vortex ring may be (numerically) stable, provided that its outer radius is very large [9]. Vortex rings in the 3D version of the CQ model (*spinning light bullets*) have also been studied in detail [10,11].

While the stability of VS's in 2D (and 3D) continuum models is an issue, the possibility of their *existence* is obvious, as a 2D complex solution can always be sought for in

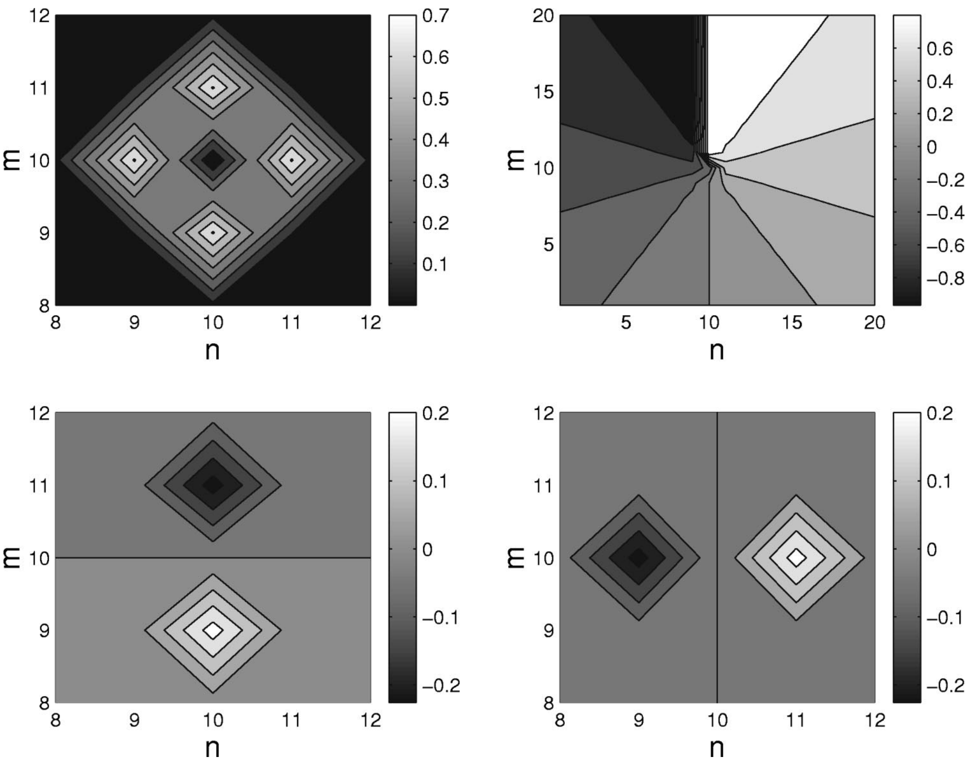

FIG. 1. An example of a stationary fundamental vortex soliton ($S=1$) in the 2D discrete NLS equation for $C=0.05$. The top left and right panels show, respectively, the absolute value and phase (in units of π) of $U_{m,n}$. The bottom left and right panels display the real and imaginary parts of the solution. All the plots show the relevant quantities in terms of gray-scale contour plots. It should be noted that in this figure, as well as in Figs. 2 and 4, the lines on top of the gray scale are a mere artifact of the plotting program (denoting points of the field with equal concentration where the color map shading changes).

the form $u=U(r)\exp(i\Lambda t+iS\theta)$, where r and θ are polar coordinates, $U(r)$ is a real amplitude, and the integer vorticity (''spin'') S is a topological invariant. The isotropy of continuous models gives rise to conservation of angular momentum, the value of which for VS's is proportional to S [11]. On the contrary, the very existence of vortex solitons in discrete models, with their broken rotational invariance, is a nontrivial issue. Of course, dynamical stability of discrete VS's, if any, will be a nontrivial problem too. The results presented below show that VS's with different values of vorticity do exist in the 2D DNLS equation, the *fundamental* one, with $S=1$, being stable in a finite parameter range, while higher-order VS's with $S\geqslant 2$ are always unstable. Notice that this result is reminiscent of the stability of 2D finite-background (''dark'') vortices in the 2D continuous NLS [12] and complex Ginzburg-Landau equations [13]. Thus, a qualitative conclusion is that, while vorticity is no longer a topological invariant in 2D lattice models, it may be, instead, a *dynamical invariant*. By this term, we mean that the time evolution of the lattice dynamics preserves the vorticity present in the configuration.

Numerical solution of the stationary equation (2), aimed at a search for a soliton of a given type by means of Newton iterations, requires an appropriate initial *Ansatz*. If one is looking for a soliton without vorticity, an appropriate *Ansatz* that can be used in discrete (as well as in continuous) models is often provided for by the variational approximation; see, e.g., Refs. [14] and [3]. However, it is not obvious how to select an appropriate vorticity-carrying *Ansatz* in the 2D lattice. Rather than trying to emulate the continuous function $\exp(iS\theta)$ which accounts for the vorticity, we used a simpler and more (computationally) robust initial *Ansatz* in the case of odd S. This choice preserves a fundamental property of VS's which is obvious in continuous models: separating the real and imaginary parts of the solution, $\mathrm{Re}\,u\sim\cos(S\theta)$ and $\mathrm{Im}\ u\sim\sin(S\theta)$, one concludes that they are odd functions of the Cartesian coordinates, respectively, x and y. This suggests that, in the discrete system, Re u and Im u must be, respectively, odd functions of the lattice coordinates m and n. In this connection, it is necessary to mention that the DNLS equation in 1D indeed gives rise to *odd* soliton solutions (with $u_{-n}=-u_n$), which have no counterpart in the continuum limit. Those solutions were investigated in detail in Refs. [5,15] under the name of ''twisted localized modes.'' In fact, they may be regarded as bound states of two strongly overlapping fundamental (even) solitons with a phase difference π between them. As has recently been demonstrated [16], only bound states of this type may be stable in 1D lattices (while bound states with the phase difference 0 are always unstable).

We were able to find fundamental ($S=1$) stationary vortex solitons in the 2D lattice governed by Eq. (2), starting with a ''dual-twisted'' *Ansatz* that was a superposition of localized *Ansätze* for $\mathrm{Re}\,u_{m,n}$ and $\mathrm{Im}\,u_{m,n}$, ''twisted'' so that $\mathrm{Re}\,u_{-m,n}=-\mathrm{Re}\,u_{m,n}$ and $\mathrm{Im}\,u_{m,-n}=-\mathrm{Im}\,u_{m,n}$. For the numerical solution of Eq. (2), the Newton methods and iterative ones based on treating the equation as a nonlinear eigenvalue problem [4] were implemented. Note, however, an essential difference between the cases $S=0$ considered in [4] and $S=1$: in the former case, $U_{m,n}$ are real, while the vorticity necessarily makes the solution complex.

Irrespective of other details, the iteration procedure which started from the ''dual-twisted'' initial *Ansatz* readily converged to a soliton which clearly kept the initial vorticity, $S=1$. A typical example of the thus obtained fundamental VS is shown in Fig. 1. Starting from a more sophisticated *An-*

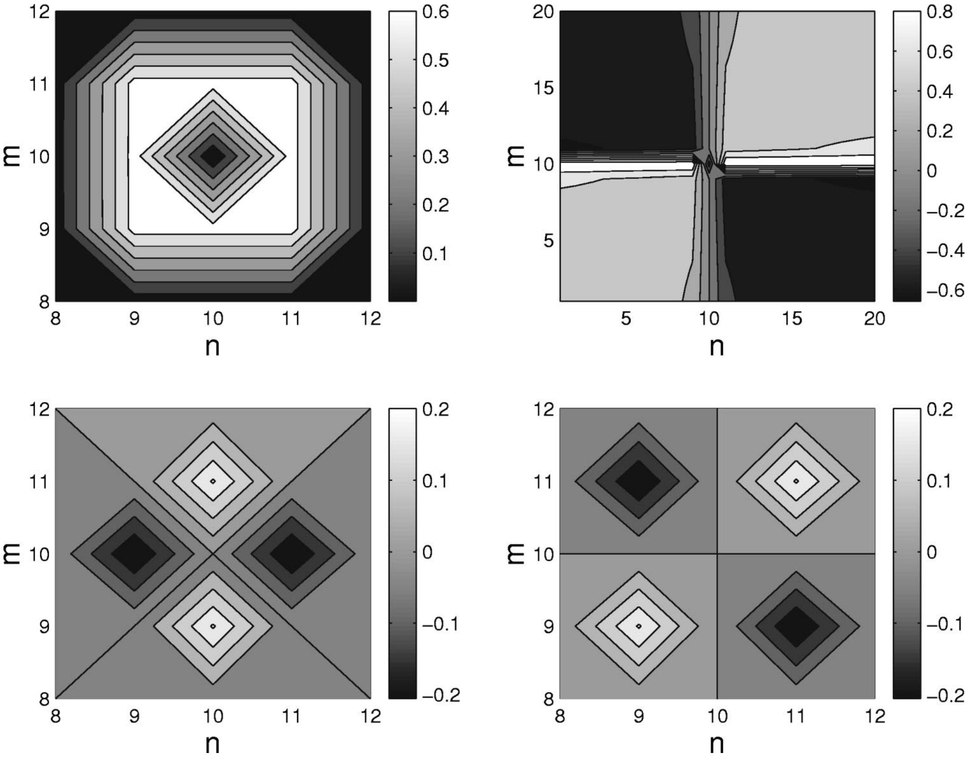

FIG. 2. An example of a stationary vortex soliton with $S=2$ for $C=0.025$. The four panels show the same quantities as in Fig. 1.

satz, it was also possible to produce stationary double vortices, i.e., solitons with $S=2$, a typical example of which can be seen in Fig. 2. An appropriate initial *Ansatz* for the $S=2$ VS's could be guessed, taking into account the even and odd parities of the $S=2$ vortices with respect to rotations, respectively, through π and $\pi/2$. The *Ansatz* which initiated iterations converging to the discrete VS with $S=2$, displayed in Fig. 2, had nonzero values of $\mathrm{Re}\, u_{m,n}$ solely along the m and n axes, and $\mathrm{Im}\, u_{m,n}$ was different from zero along the diagonals $n=\pm m$.

The next issue is the stability of VS's. Our stability investigation is based on linear stability analysis around the stationary solution, complemented by direct simulations of the evolution of perturbed solitons. For the linear stability analysis, a perturbed solution was taken as $u_{m,n}=\exp(i\Lambda t)[U_{m,n}+a_{m,n}\exp(-i\omega t)+b_{m,n}\exp(i\omega^* t)]$, ω being an eigenvalue. The substitution of this into Eq. (1) and linearization in the infinitesimal perturbation amplitudes $a_{m,n}$ and $b_{m,n}$ leads to a system of linear homogeneous equations, the computation of the eigenvalues amounting to numerical diagonalization of the corresponding matrix.

Instability is accounted for by eigenvalues with a nonzero imaginary part. Because of the Hamiltonian nature of DNLS, complex eigenvalues may appear in conjugate pairs or quartets, $|\mathrm{Im}\,\omega|$ being the instability gain. It has been found that the stability of the $S=1$ VS depends on the value of the coupling constant C, while the $S=2$ solitons are *always* linearly unstable.

It is obvious that all VS's, including the fundamental ones with $S=1$, must be unstable at C sufficiently large, as $C\to\infty$ implies a transition to the continuous 2D NLS equation, in which any soliton is unstable. In accordance with this, our computations show that, with the increase of C, the $S=1$ VS becomes unstable at $C=C_{\mathrm{cr}}^{(1)}\approx 0.13$, and it remains unstable for $C>C_{\mathrm{cr}}^{(1)}$. In terms of the eigenvalues, a destabilizing bifurcation happens when a pair of purely real isolated eigenvalues, which bifurcated at a smaller value of C from the edge of the phonon eigenvalue band, collides with another isolated pair of real eigenvalues which originally existed close to the origin but was gradually approaching the band. The collision results in the appearance of a complex eigenvalue quartet, i.e., instability. An example of the spectrum in the complex plane of the eigenvalues, containing the unstable quartet, is shown in Fig. 3. It should be noted that collisions inducing the generation of quartets of eigenvalues are expected, since even for 1D ‘‘twisted’’ solitons, collisions of isolated eigenvalues with the band eigenvalues of an (apparently) opposite Krein sign lead to such *Hopf-like* bifurcations [15].

The usual $(S=0)$ soliton must also become unstable with the increase of C, as its counterpart is unstable too in the continuum limit [4]. It is natural to compare the instabilities of the vortex and usual solitons on the lattice. We have found that, at $C\approx 0.15$, a pair of isolated real (stable) eigenvalues $\pm|\omega|$ splits off from the phonon band in the spectrum of the $S=0$ soliton. A destabilizing bifurcation takes place at $C=C_{\mathrm{cr}}^{(0)}\approx 0.32$: the pair hits the origin and reappears in a purely imaginary form, $\pm i|\omega|$. As concerns the $S=2$ soliton, its eigenvalue spectrum always contains at least one purely imaginary pair.

The shape of eigenmodes related to unstable eigenvalues is important too, as it determines the actual type of a perturbation that is going to destroy the soliton. Computations show that the unstable eigenmode of the $S=1$ soliton, if any, has a shape breaking the (discrete) symmetry of the unper-

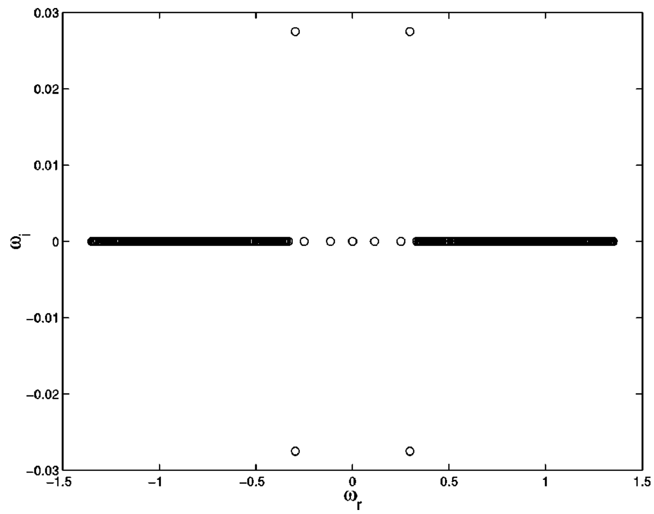

FIG. 3. Eigenvalues ω of infinitesimal perturbations around the unstable fundamental vortex soliton with $S=1$ at $C\approx 0.13$, just above the onset of the instability. Shown is the spectral plane of the complex (ω_i) versus the real (ω_r) part of the eigenvalue.

turbed soliton so as to split it into two parts. This is quite similar to the unstable eigenmode of the $S=1$ VS's in the 2D continuous NLS equation with CQ nonlinearity [11]. However, a drastic difference is that all VS's, including the fundamental ones with $S=1$, are unstable in the latter model (although the instability may be very weak if the VS is very broad), while in the discrete model (which contains no quintic nonlinearity) there is a well-defined stability region for the VS's with $S=1$.

The stability of the VS's with $S=1$ at $C<C_{\mathrm{cr}}^{(1)}\approx 0.13$, and their instability at $C>C_{\mathrm{cr}}^{(1)}$ was also verified by direct simulations of the full nonlinear equation (1), which used a fourth-order Runge-Kutta integrator. In the former case, the solitons always remain unscathed for very long integration times. In the latter case, the growing instability at first splits the VS into two usual ($S=0$) solitons (see Fig. 4), which strongly resembles the development of the above-mentioned instability in the continuous CQ model. However, further development is quite different. As is seen in Fig. 4, in the course of long evolution one of the secondary $S=0$ solitons eventually dies, decaying into lattice phonons, while the other one survives, which may be regarded as an instability-induced spontaneous symmetry breaking. Thus, instead of two separating $S=0$ solitons, which is a generic outcome of the instability in the 2D continuous CQ model [11], in the discrete model we end up with a single quiescent $S=0$ soliton (which is corroborated by many other runs of the direct simulations). This outcome is possible because the discrete systems do not conserve angular momentum.

The outcome of the instability development for $S=2$ VS's was also found by means of simulations. Eigenmodes related to the above-mentioned pair of imaginary eigenvalues, which are amenable to the instability of the $S=2$ soliton, have a shape (not shown here) suggesting that they will initiate cleaving the soliton in two. Direct simulations comply with this expectation, demonstrating that the unstable $S=2$ VS splits into two $S=1$ solitons. However, the thus-generated *nonstationary* $S=1$ vortices, unlike their stationary counterparts, may be subject to a weak instability. Our simulations demonstrate that they eventually break up into complexes of $S=0$ solitons. In this connection, it is relevant to recall that, in the 2D continuous CQ model, $S=2$ VS's split into four $S=0$ solitons which fly out in tangential directions [11].

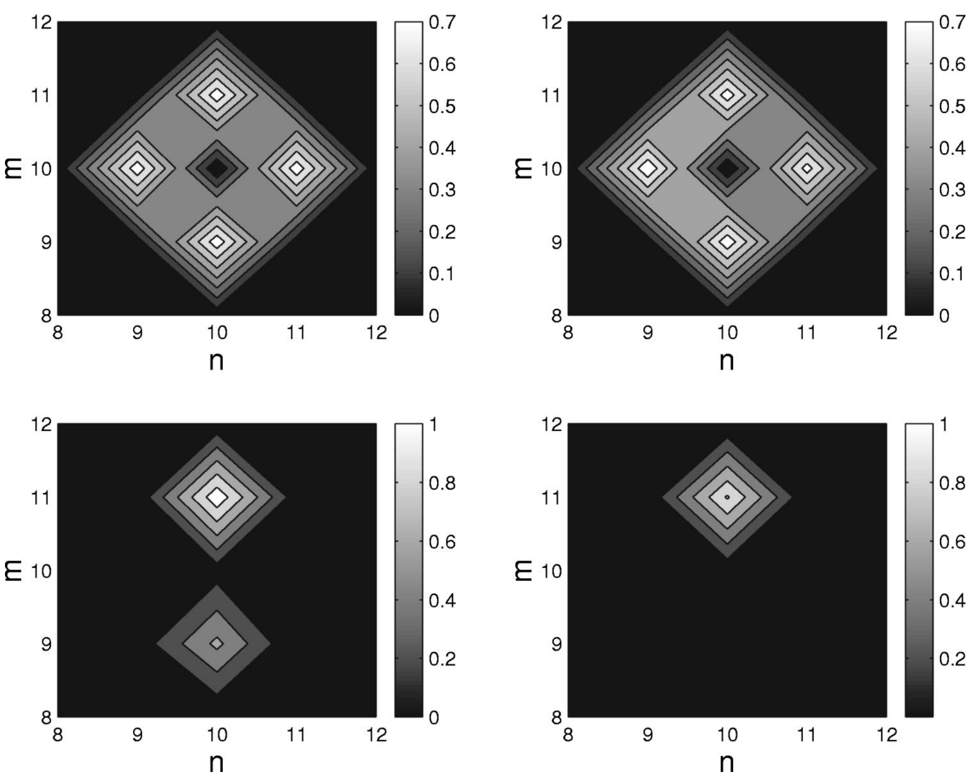

FIG. 4. Time evolution of the unstable $S=1$ vortex soliton. The panels show snapshots of the evolution in terms of gray-scale contour plots of $|U_{m,n}|$. The initial configuration is not shown, as it is very similar to that of Fig. 1. The first snapshot is taken at $t=420$ [time is measured in time units of Eq. (1)], and the time interval between the snapshots is $\Delta t=100$. The initial breakup of the the $S=1$ vector soliton into two $S=0$ solitons, and subsequent breaking of the symmetry between them, ending up with the survival of a single $S=0$ soliton, can be observed. The initial condition consisted of a small random perturbation on top of the exact $S=1$ soliton.

An additional important difference between the $S=0$ and $S=1$ discrete solitons should be highlighted here. As is well known [17–19], when bistability is present for the $S=0$ modes, unstable (''broad'') solitary waves coexist with stable (''narrow'') ones. On the contrary, the dynamics of vortex-like discrete solitons resemble those of their one-dimensional counterpart, namely ''twisted localized modes.'' In particular, it is well known [15,16] that the component pulses in the real and imaginary parts of such twisted modes repel each other [see, e.g., Fig. 3 in Ref. [15] and Fig. 1(b) in Ref. [16]]. Therefore, a discrete vortex with maxima of the (absolute value of the) real part at the lattice points $(m-1,n),(m+1,n)$ and of the imaginary part at $(m,n+1),(m,n-1)$ has larger energy than the vortex with maxima of the (absolute values of the) real and imaginary parts, respectively, at the points $(m-2,n),(m+2,n)$ and $(m,n+2),(m,n-2)$. Hence, ''broader'' discrete-vortex solutions are more energetically favorable in this case.

The mode created here is by construction (i.e., by virtue of the selection of the initial condition) one that is centered on a lattice site. However as can be seen in Refs. [4,20], an additional type of a mode that is feasible for $S=0$ solitons is a so-called Page mode, namely, the one centered between lattice sites in both directions. A similar configuration can also be constructed in the case of a vortex. In particular, we have constructed such vortex solitons by centering maxima (of the absolute values) of the ''dual-twisted'' initial *Ansatz* at the points $(m,n),(m+1,n+1)$ for the real part and at $(m+1,n),(m,n+1)$ for the imaginary part of the solution [as opposed to centering them at $(m-1,n),(m+1,n)$ and $(m,n+1),(m,n-1)$, respectively, in the solutions considered above]. In agreement with the arguments mentioned above, we have found that this Page-like mode has a higher energy; hence, it is a less stable configuration. For example, for $C=0.0125$, the mode with absolute maxima of the dual-twisted *Ansatz* at $(m-1,n),(m+1,n)$ for $\mathrm{Re}(u_{m,n})$ and at $(m,n+1),(m,n-1)$ for $\mathrm{Im}(u_{m,n})$ had energy $E=-0.20169$, while the one with absolute maxima at $(m-2,n),(m+2,n)$ for $\mathrm{Re}(u_{m,n})$ and at $(m,n+2),(m,n-2)$ for $\mathrm{Im}(u_{m,n})$ had $E=-0.20232$ and the one with absolute maxima at $(m,n),(m+1,n+1)$ for $\mathrm{Re}(u_{m,n})$ and at $(m,n+1),(m+1,n)$ for $\mathrm{Im}(u_{m,n})$ had $E=-0.20106$. Three-dimensional plots of the absolute value of the Page mode as well as of its real and imaginary part are given in Fig. 5. It should also be noted that, as the ''broader'' configurations are of lower (equilibrium) energy, the narrow ones, if appropriately perturbed, can rearrange themselves into broad states, shedding the energy difference into kinetic energy, as can be seen in Fig. 6 of Ref. [16].

Last, since the $S=1$ and $S=0$ solitons may coexist, at $C<C_{\mathrm{cr}}^{(1)}$, as stable solutions of the 2D DNLS equation, it is natural to consider interactions between them (as well as between two like solitons). Detailed analysis of the interactions between solitons and their possible bound states in the present model will be presented elsewhere. Here, we just mention an essential finding: a usual soliton with $S=0$ and a fundamental VS with $S=1$ can readily form a stable bound state.

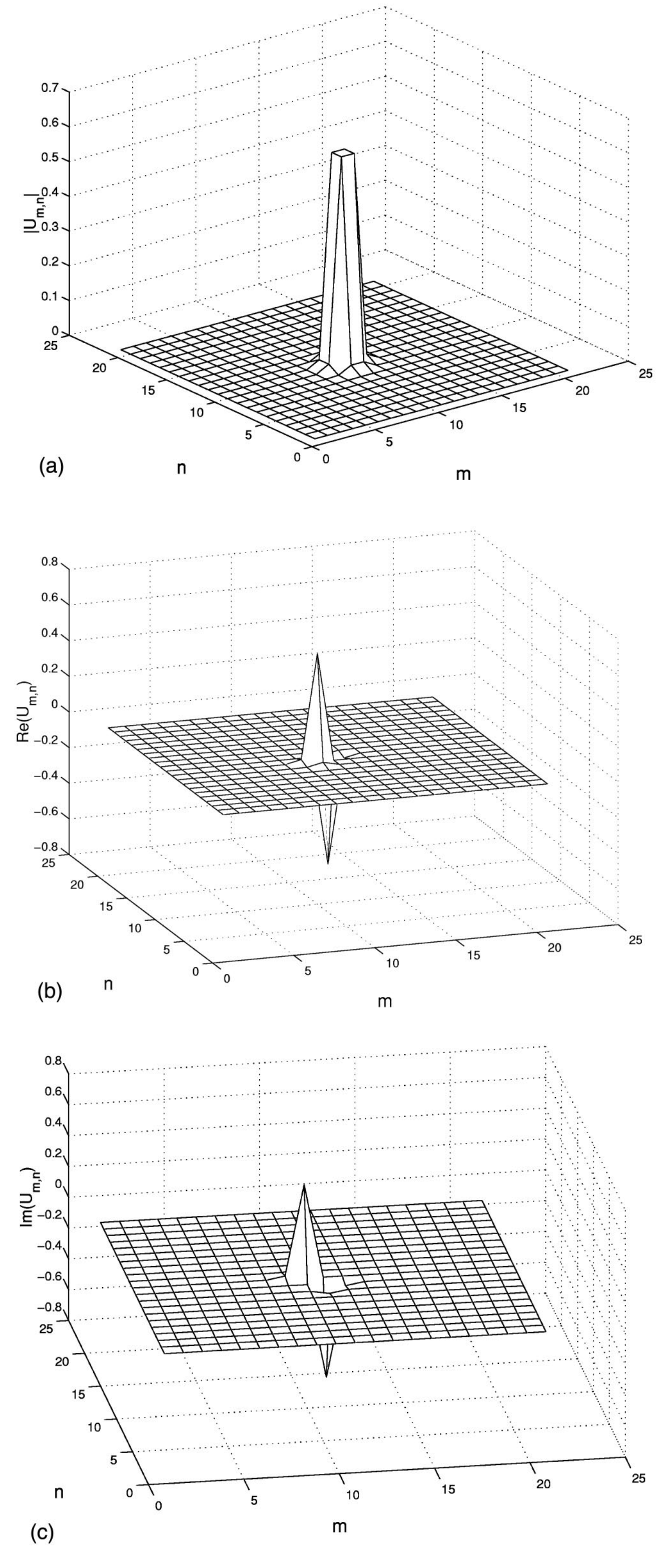

FIG. 5. Three-dimensional profile of the absolute value (a), real part (b), and imaginary part (c), of a Page-like mode for $C=0.0125$.

In conclusion, we have found localized states in two-dimensional nonlinear dynamical lattices, in the form of vortex solitons with an integer vorticity S. While Hamiltonian lattices do not conserve angular momentum the topological charge related to it, our results show that vorticity may be conserved as a dynamical, rather than topological, invariant. The fundamental vortex solitons with $S=1$ are completely stable if the intersite coupling constant C is smaller than a critical value $C_{\rm cr}^{(1)}$. At $C>C_{\rm cr}^{(1)}$, an instability appears through a quartet of complex eigenvalues in the linearized equations. Direct simulations demonstrate that, if the vortex soliton is unstable, it first splits into two usual solitons with $S=0$ in accordance with the prediction of the linear analysis; then, one of them decays into radiation, while the other one survives. We have also demonstrated that the $S=0$ solitons become unstable at $C>C_{\rm cr}^{(0)}$, with $C_{\rm cr}^{(0)}/C_{\rm cr}^{(1)}\approx 2.46$, and their route to instability is different from that for $S=1$, being accounted for by a pair of imaginary eigenvalues. Except for the lower energy centered on site-discrete vortex solitons, also higher-energy, Page-like ones centered between sites were constructed. Vortex solitons with $S=2$ were found too, but they are always unstable.

The most realistic possibility to observe the vortex soliton predicted in this work is provided by a bunch of optical fibers forming a rectangular lattice in the plane normal to the fibers. In principle, the vortex soliton in the latter system may also have a potential for the optical storage of data. In that context, it may be quite important that there be a stable localized configuration principally different from the ordinary (zero-vorticity) soliton.

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